Gaussian Integer Continued Fractions

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30th April 2015
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Integer Continued Fractions

Definition

A finite *integer continued fraction* is a continued fraction of the form

\[
[b_1, b_2, b_3, \ldots, b_n] = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots - \frac{1}{b_n}}},
\]

where \(b_i \in \mathbb{Z}\) for \(i = 1, 2, \ldots, n\).
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where \(b_i \in \mathbb{Z}\) for \(i = 1, 2, \ldots, n\).

An infinite integer continued fraction is defined to be the limit

\[
[b_1, b_2, \ldots] = \lim_{i \to \infty} [b_1, b_2, \ldots, b_i],
\]

of its sequence of convergents.
The Modular group

The Modular group, $\Gamma$, is the group generated by the Möbius maps

$$S(z) = z + 1 \quad \text{and} \quad T(z) = -\frac{1}{z}.$$
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The elements of \( \Gamma \) are the Möbius maps

\[
z \mapsto \frac{az + b}{cz + d}
\]

with \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \). They are isometries of the hyperbolic upper half-plane \( \mathbb{H} \).
The Modular group and continued fractions

Any element of $\Gamma$ can be written $S^{b_1} TS^{b_2} T \ldots S^{b_n} TS^{b_{n+1}}$. 
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$$S^{b_1} T S^{b_2} T \ldots S^{b_n} T S^{b_{n+1}}(\infty) = b_1$$
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S^{b_1} TS^{b_2} T \ldots S^{b_n} TS^{b_{n+1}}(\infty) = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots - \frac{1}{b_n}}}
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Conversely, $[b_1, b_2, \ldots, b_n]$ can be associated the word $S^{b_1} TS^{b_2} T \ldots S^{b_n} T$. 
The Modular group and continued fractions

Any element of $\Gamma$ can be written $S^{b_1} T S^{b_2} T \ldots S^{b_n} T S^{b_{n+1}}$.

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Conversely, $[b_1, b_2, \ldots, b_n]$ can be associated the word $S^{b_1} T S^{b_2} T \ldots S^{b_n} T$.

There is a ‘bijection’ between elements of $\Gamma$ and finite integer continued fractions.
The Farey graph

We work in the hyperbolic upper half-plane $\mathbb{H}$. Let $L$ denote the line segment joining 0 to $\infty$.

The Farey graph, $\mathcal{F}$, is the orbit of $L$ under $\Gamma$: Edges are images of $L$, and vertices are images of $\infty$, under elements of $\Gamma$. 
The Farey graph

$\mathcal{F}$ is an infinite graph.

Vertices lie on $\mathbb{R} \cup \{\infty\}$. They are precisely the rational numbers, plus $\infty$. Each vertex has infinite valency. The vertices neighbouring $\infty$ are the integers.
Paths in the Farey graph

\[ [b_1, b_2, \ldots, b_n] = S^{b_1} T S^{b_2} T \ldots S^{b_n} T(\infty) \]

so finite continued fractions are vertices of \( \mathcal{F} \). In particular, each convergent \( C_i \) of an infinite continued fraction is a vertex of \( \mathcal{F} \).
[b_1, b_2, \ldots, b_n] = S^{b_1} T S^{b_2} T \ldots S^{b_n} T(\infty)

so finite continued fractions are vertices of \( F \). In particular, each convergent \( C_i \) of an infinite continued fraction is a vertex of \( F \).

**Theorem**

A sequence of vertices \( \infty = v_1, v_2, v_3 \ldots \) forms an infinite path in \( F \) if and only if they are the consecutive convergents of an infinite integer continued fraction expansion.
Paths in the Farey graph

\[[b_1, b_2, \ldots, b_n] = S^{b_1} T S^{b_2} T \ldots S^{b_n} T(\infty)\]

so finite continued fractions are vertices of \(\mathcal{F}\). In particular, each convergent \(C_i\) of an infinite continued fraction is a vertex of \(\mathcal{F}\).

**Theorem**

A sequence of vertices \(\infty = v_1, v_2, v_3 \ldots\) forms an infinite path in \(\mathcal{F}\) if and only if they are the consecutive convergents of an infinite integer continued fraction expansion.

There is a ‘bijection’ between integer continued fractions and paths in the Farey graph with initial vertex \(\infty\).
Paths in the Farey graph

Take, for example, \([0, -2, 1, 3, \ldots]\)

\[C_1 = 0, \quad C_2 = \frac{1}{2}, \quad C_3 = \frac{1}{3}, \quad C_4 = \frac{2}{7}, \ldots\]
The geometry of integer continued fractions

We can reformulate questions about continued fractions into questions about paths.

- The theory of simple continued fractions can be developed.
The geometry of integer continued fractions

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- The theory of simple continued fractions can be developed.
- An integer continued fraction is geodesic if and only if its corresponding path in the Farey graph is.
The geometry of integer continued fractions

We can reformulate questions about continued fractions into questions about paths.

- The theory of simple continued fractions can be developed.
- An integer continued fraction is geodesic if and only if its corresponding path in the Farey graph is.
- An infinite integer continued fraction converges if and only if its corresponding path in the Farey graph converges.
We can reformulate questions about continued fractions into questions about paths.

- The theory of simple continued fractions can be developed.
- An integer continued fraction is geodesic if and only if its corresponding path in the Farey graph is.
- An infinite integer continued fraction converges if and only if its corresponding path in the Farey graph converges.

Can we develop a similar theory for other classes of continued fractions?
Gaussian Integer Continued Fractions

Definition

A finite *Gaussian integer continued fraction* is a continued fraction of the form

\[
[b_1, b_2, b_3, \ldots, b_n] = b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots + \frac{1}{b_n}}},
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where \( a_i \in \mathbb{Z}[i] \) for \( i = 1, 2, \ldots, n \).
Gaussian Integer Continued Fractions

Definition

A finite Gaussian integer continued fraction is a continued fraction of the form

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where \( a_i \in \mathbb{Z}[i] \) for \( i = 1, 2, \ldots, n \).

An infinite Gaussian integer continued fraction is defined to be the limit

\[ [b_1, b_2, \ldots] = \lim_{i \to \infty} [b_1, b_2, \ldots, b_i] \]

of its sequence of convergents.
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The Picard group, $P$, is the group generated by the Möbius maps

$$R(z) = z + i \quad \text{and} \quad S(z) = z + 1 \quad \text{and} \quad T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = \frac{1}{z}.$$
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The elements of $P$ are the Möbius maps

$$z \mapsto \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{Z}[i]$ and $|ad - bc| = 1$. They can be extended via the Poincaré extension to isometries of the hyperbolic upper half-space $\mathbb{H}^3$. 
The Picard group and continued fractions

Given $w = a + bi \in \mathbb{Z}[i]$ let $S_w = S^a R^b$ so that $S_w(z) = z + a + bi$.

Any element of $P$ can be written $S_{b_1} US_{b_2} U \ldots S_{b_n} US_{b_{n+1}}$ where each $b_i \in \mathbb{Z}[i]$.
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Conversely, \([b_1, b_2, \ldots, b_n]\) can be associated the word \( S_{b_1} U S_{b_2} U \ldots U S_{b_n} \).
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Conversely, \([b_1, b_2, \ldots, b_n]\) can be associated the word \( S_{b_1} U S_{b_2} U \ldots U S_{b_n} \).

There is a ‘bijection’ between elements of \( P \) and finite Gaussian integer continued fractions.
The Picard-Farey Graph

Definition

The Picard-Farey graph, $\mathcal{G}$, is formed as the orbit of the vertical line segment $L$ with endpoints 0 and $\infty$ under the Picard group.
The Picard-Farey Graph

**Definition**

The *Picard-Farey graph*, $\mathcal{G}$, is formed as the orbit of the vertical line segment $L$ with endpoints 0 and $\infty$ under the Picard group.

It is a three-dimensional analogue of the Farey graph.

![Graph Image](image-url)
Properties of the Picard-Farey graph

- The Picard-Farey graph is the 1-skeleton of a tessellation of $\mathbb{H}^3$ by ideal hyperbolic octahedra.
- The vertices $V(G)$ are of the form $\frac{a}{c}$ with $a, c \in \mathbb{Z}[i]$: they are precisely those complex numbers with rational real and complex parts, and $\infty$ itself.
- The edges of $G$ are hyperbolic geodesics. Two vertices $\frac{a}{c}$ and $\frac{b}{d}$ are neighbours - joined by an edge - in $G$ if and only if $|ad - bc| = 1$.
- Elements of $P$ are graph automorphisms of $G$. 
Paths in the Picard-Farey graph

\[ [b_1, b_2, \ldots, b_n] = S_{b_1} U S_{b_2} U \ldots U S_{b_n} T(\infty) \]

so finite Gaussian integer continued fractions are vertices of \( \mathcal{G} \). In particular, each convergent \( C_i \) of an infinite Gaussian integer continued fraction is a vertex of \( \mathcal{G} \).
Paths in the Picard-Farey graph

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so finite Gaussian integer continued fractions are vertices of \( G \). In particular, each convergent \( C_i \) of an infinite Gaussian integer continued fraction is a vertex of \( G \).

**Theorem**

A sequence of vertices \( \infty = v_1, v_2, v_3 \ldots \) forms an infinite path in \( G \) if and only if they are the consecutive convergents of an infinite Gaussian integer continued fraction expansion.
Paths in the Picard-Farey graph

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**Theorem**

A sequence of vertices \( \infty = v_1, v_2, v_3 \ldots \) forms an infinite path in \( G \) if and only if they are the consecutive convergents of an infinite Gaussian integer continued fraction expansion.

There is a ‘bijection’ between Gaussian integer continued fractions and paths in the Picard-Farey graph with initial vertex \( \infty \).
Convergence of continued fractions

An integer continued fraction is simple if for all \( i > 1, b_i > 0 \). Every infinite simple continued fraction converges.
Convergence of continued fractions

An integer continued fraction is simple if for all $i > 1$, $b_i > 0$. Every infinite simple continued fraction converges.

What about general Gaussian integer continued fractions?

$$[0, 1, -1, 1, -1, \ldots] = 0 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}$$
Convergence of continued fractions

An integer continued fraction is simple if for all $i > 1$, $b_i > 0$. Every infinite simple continued fraction converges.

What about general Gaussian integer continued fractions?

$$[0, 1, -1, 1, -1, \ldots] = 0 + \frac{1}{1 + \frac{1}{-1 + \frac{1}{1 + \ldots}}}$$

$c_1 = 0, \quad c_2 = 1, \quad c_3 = \infty, \quad c_4 = 0, \quad c_5 = 1, \quad c_6 = \infty, \ldots$
Convergence of continued fractions

An integer continued fraction is simple if for all $i > 1$, $b_i > 0$. Every infinite simple continued fraction converges.

What about general Gaussian integer continued fractions?

$$[0, 1, -1, 1, -1, \ldots] = 0 + \frac{1}{1 + \frac{1}{1 - 1 + \frac{1}{1 + \ldots}}}$$

$c_1 = 0, \quad c_2 = 1, \quad c_3 = \infty, \quad c_4 = 0, \quad c_5 = 1, \quad c_6 = \infty, \ldots$

When does a Gaussian integer continued fraction converge?
Convergence of Gaussian Integer Continued Fractions

Literature on this topic generally restricts to certain classes of Gaussian integer continued fractions, such as those obtained using algorithms. See, for example, Dani and Nogueira [2].

Can we find a more general condition for convergence that can be applied to all Gaussian integer continued fractions?

The question

“When does a Gaussian integer continued fraction \([a_1, a_2, \ldots, a_n]\) converge?”

can be reformulated as the question

“When does an infinite path in \(G\) with initial vertex \(\infty\) converge?”
Convergence of Integer Continued Fractions

Theorem

An infinite path in $\mathcal{F}$ with vertices $\infty = v_1, v_2, v_3, \ldots$ converges to an irrational number $x$ if and only if the sequence $v_1, v_2, \ldots$ contains no constant subsequence.
Theorem

An infinite path in $\mathcal{F}$ with vertices $\infty = v_1, v_2, v_3, \ldots$ converges to an irrational number $x$ if and only if the sequence $v_1, v_2, \ldots$ contains no constant subsequence.

Proof.

$\implies$ Clear.
Convergence of Integer Continued Fractions

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**Proof.**

$\implies$ Clear.  $\iff$ Assume that $\{v_i\}$ diverges, so it has two accumulation points, $v_1$ and $v_2$. There is some edge of $\mathcal{F}$, with endpoints $u$ and $w$ ‘separating’ $v_1$ and $v_2$. 

![Diagram showing the path and points](image)
Convergence of Integer Continued Fractions

**Theorem**

An infinite path in $\mathcal{F}$ with vertices $\infty = v_1, v_2, v_3, \ldots$ converges to an irrational number $x$ if and only if the sequence $v_1, v_2, \ldots$ contains no constant subsequence.

**Proof.**

$\implies$ Clear. $\impliedby$ Assume that $\{v_i\}$ diverges, so it has two accumulation points, $v_1$ and $v_2$. There is some edge of $\mathcal{F}$, with endpoints $u$ and $w$ ‘separating’ $v_1$ and $v_2$.

Thus the path must pass through one of $u$ or $v$ infinitely many times, and has a convergent subsequence.
A Problem

The key property used here is that removing any edge of $\mathcal{F}$ separates it into two connected components.

In the Picard-Farey graph, $\mathcal{G}$, there is no such property: removing any finite number of edges will not separate $\mathcal{G}$ into two connected components.
A Problem

The key property used here is that removing any edge of $\mathcal{F}$ separates it into two connected components.

In the Picard-Farey graph, $\mathcal{G}$, there is no such property: removing any finite number of edges will not separate $\mathcal{G}$ into two connected components. Is there a ‘nice’ infinite set that we can use instead?

Along $\hat{\mathbb{R}}$ lies a vertical Farey graph.

![Vertical Farey graph]

Removing it separates $\mathcal{G}$ into two connected components.
The Real Line

Thus any path that crosses $\hat{\mathbb{R}}$ must pass through a vertex lying on the extended real line.
The Real Line

Thus any path that crosses $\hat{\mathbb{R}}$ must pass through a vertex lying on the extended real line.

Elements of the Picard group are automorphisms of $G$, so any image of $\hat{\mathbb{R}}$ has this same property.

**Definition**

A *Farey section* is an image of $\hat{\mathbb{R}}$ under an element of $G$. 
Farey Sections

Farey sections cover \( \hat{\mathbb{C}} \) densely.

Each Farey section has the property that if a path crosses it then it must pass through it.
Farey Sections

Farey sections cover \( \hat{\mathbb{C}} \) densely.

Each Farey section has the property that if a path crosses it then it must pass through it.

If a path crosses a Farey section infinitely many times, then it either has an accumulation point in that Farey section, or passes through some vertex of that Farey section infinitely many times.
Convergence of Gaussian Integer Continued Fractions

Theorem

An infinite path in $\mathcal{G}$ with vertices $\infty = v_1, v_2, v_3, \ldots$ converges to $x \notin V(\mathcal{G})$ if and only if the sequence $v_1, v_2, \ldots$ contains no constant subsequence and has only finitely many accumulation points.
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**Proof.**

$\implies$ Clear. $\iff$ Assume that $\{v_i\}$ diverges, so it has two accumulation points, $v_1$ and $v_2$. There is an infinite family of Farey sections ‘separating’ $v_1$ and $v_2$. 
Convergence of Gaussian Integer Continued Fractions

**Theorem**

An infinite path in $\mathcal{G}$ with vertices $\infty = v_1, v_2, v_3, \ldots$ converges to $x \notin V(\mathcal{G})$ if and only if the sequence $v_1, v_2, \ldots$ contains no constant subsequence and has only finitely many accumulation points.

**Proof.**

$\implies$ Clear. $\iff$ Assume that $\{v_i\}$ diverges, so it has two accumulation points, $v_1$ and $v_2$. There is an infinite family of Farey sections ‘separating’ $v_1$ and $v_2$.

$v_i$ either has an accumulation point on each Farey section, or has a constant subsequence.
Examples

Do we need the added condition? Can we say that an infinite path in $G$ with vertices $\infty = v_1, v_2, v_3, \ldots$ converges to $x \not\in V(G)$ if and only if the sequence $v_1, v_2, \ldots$ contains no constant subsequence?
Examples

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Lemma

There exist paths with no constant subsequence that do not converge.

Proof.

Given $z \neq w$, choose sequences $z_i \to z$ and $w_i \to w$. Because removing finitely many edges does not disconnect $\mathcal{G}$, we can construct a simple path that passes through each $z_i$ and $w_i$, and thus has both $z$ and $w$ as accumulation points.
Summary

To summarise:

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- Gaussian integer continued fractions can be viewed as paths in the Picard-Farey graph.
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Where next?

- What else can we say about Gaussian integer continued fractions using the Picard-Farey graph?
- Can we use hyperbolic geometry to study the continued fractions associated to other types of composition sequences?
Thanks for listening!

:)
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