Strong approximation in Fuchsian groups: A geometric approach

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The thrice-punctured sphere

The thrice-punctured sphere is the sphere with three points removed.

The thrice-punctured sphere is the largest subdomain of \( \hat{\mathbb{C}} \) supporting a hyperbolic metric.
The group $\Gamma(2)$

The *modular group*, $\Gamma$, is the group of Möbius transformations

$$z \mapsto \frac{az + b}{cz + d},$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. 
The group $\Gamma(2)$

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The principal congruence subgroup of $\Gamma$ of level two, $\Gamma(2)$, is the subgroup of $\Gamma$ with elements satisfying

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \pmod{2}. $$
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$\Gamma(2)$ is a Fuchsian group: It is a discrete group of isometries of the hyperbolic upper half-plane $\mathbb{H}$. 

The thrice-punctured sphere and continued fractions
Obtaining the thrice-punctured sphere

The region $D$ is a fundamental domain for the action of $\Gamma(2)$ on $\mathbb{H}$.

Side-pairing transformations are

$$A(z) = z + 2 \quad \text{and} \quad B(z) = -\frac{z}{2z - 1}.$$ 

The quotient $\mathbb{H}/\Gamma(2)$ is a sphere with three cusps, at 0, 1 and $\infty$. 
A tessellation of $\mathbb{H}$

The images of $D$ under $\Gamma(2)$ tessellate $\mathbb{H}$.
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Vertices are the orbits of the cusps 0, 1, and $\infty$ under $\Gamma(2)$. They are the rational numbers and $\infty$. 
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Vertices are the orbits of the cusps 0, 1, and $\infty$ under $\Gamma(2)$. They are the rational numbers and $\infty$.

We can associate to each irrational number $x$ a unique sequence of $A$s and $B$s.
An example

\[ x = \lim_{n \to \infty} (AB^2)^n(\infty). \]
**Even-integer continued fractions**

### Definition

A finite *even-integer continued fraction* is a continued fraction of the form

\[
[2b_1, 2b_2, \ldots, 2b_n] = 2b_1 - \frac{1}{2b_2 - \frac{1}{2b_3 - \cdots - \frac{1}{2b_n}}}.
\]

where \(b_1 \in \mathbb{Z}\) and \(b_2, b_3, \ldots, b_n \in \mathbb{Z} \setminus \{0\}$. 

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Even-integer continued fractions

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\]

where \(b_1 \in \mathbb{Z}\) and \(b_2, b_3, \ldots, b_n \in \mathbb{Z} \setminus \{0\}\).

Given an infinite sequence \(b_1, b_2, \ldots\) we can define an infinite even-integer continued fraction \([2b_1, 2b_2, \ldots]\).

We define the convergents \(C_k = [2b_1, \ldots, 2b_k]\) of \([2b_1, 2b_2, \ldots]\). If this sequence converges to a limit \(x\), then we say that \([2b_1, 2b_2, \ldots]\) is an even-integer continued fraction expansion of \(x\).
Even-integer continued fractions

Given an irrational number, consider its sequence $A^{b_1} B^{b_2} A^{b_3} \ldots$.

\[ A^n(z) = 2n + z \quad \text{and} \quad B^n(z) = -\frac{1}{2n - \frac{1}{z}} \]

So

\[ A^{b_1} B^{b_2} A^{b_3} \ldots B^{b_k}(\infty) = 2b_1 - \frac{1}{2b_2 - \frac{1}{2b_3 - \cdots - \frac{1}{2b_k}}} \]

Any irrational number has a unique infinite even-integer continued fraction expansion.
Geodesics

Geodesics on $\mathbb{H}/\Gamma(2)$ lift to geodesics in $\mathbb{H}$.

The different lifts form an equivalence class under the action of $\Gamma(2)$. 
Coding geodesics

Providing the geodesic on $\mathbb{H}/\Gamma(2)$ doesn’t start or end at a cusp, the endpoints of a lift will be irrational, so have unique even-integer continued fraction expansions.

Coding geodesics as even-integer continued fractions is particularly useful for studying closed geodesics on $\mathbb{H}/\Gamma(2)$. 
ECFs in the literature

One of the first papers treating even-integer continued fractions (ECFs) systematically was that of Kraaikamp and Lopes [3], who used ECFs to study the lengths of closed geodesics.
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They showed that:

- ECFs always converge, and a real number has a unique ECF expansion unless it is a face-centre point.
- Two ECF expansions are equivalent if and only if they have the same tail.
- An irrational number has an eventually periodic ECF expansion if and only if it is a quadratic irrational.
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These results are analogous to ones for regular continued fractions. What’s missing is results on Diophantine approximation.
Strong approximation

Definition

A rational number $\frac{a}{b}$ is a strong approximant for an irrational number $\alpha$ if for any other rational number $\frac{c}{d}$ with $d \leq b$,

$$|b\alpha - a| \leq |d\alpha - c|,$$

with equality if and only if $\frac{a}{b} = \frac{c}{d}$.
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\[ |b\alpha - a| \leq |d\alpha - c|, \]

with equality if and only if \( \frac{a}{b} = \frac{c}{d} \).

We have the following classical result:

Theorem
A non-integer rational number is a strong approximant for an irrational number \( \alpha \) if and only if it is a convergent of the regular continued fraction expansion of \( \alpha \).
ECF convergents as strong approximants

ECF convergents do not correspond to strong approximants.

Let $P$ be the set of rational numbers $\frac{p}{q}$ with either $p$ odd and $q$ even, or $p$ even and $q$ odd.
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Let $P$ be the set of rational numbers $\frac{p}{q}$ with either $p$ odd and $q$ even, or $p$ even and $q$ odd.

**Definition**

A rational number $\frac{a}{b} \in P$ is a *theta-strong approximant* for an irrational number $\alpha$ if for any other rational number $\frac{c}{d} \in P$ with $d \leq b$,

$$|b\alpha - a| \leq |d\alpha - c|,$$

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with equality if and only if $\frac{a}{b} = \frac{c}{d}$.

**Theorem**

A rational number is a theta-strong approximant for an irrational number $\alpha$ if and only if it is a convergent of the ECF expansion of $\alpha$. 
The theta group

We work with a slightly different group $\Gamma(2)$. The *theta group*, $\Gamma_\theta$, is the group generated by the Möbius transformations

\[
S(z) = z + 2 \quad \text{and} \quad T(z) = -\frac{1}{z}
\]
The theta group

We work with a slightly different group $\Gamma(2)$. The theta group, $\Gamma_\theta$, is the group generated by the Möbius transformations

$$S(z) = z + 2 \quad \text{and} \quad T(z) = -\frac{1}{z}$$

The theta group consists of elements of the form

$$z \rightarrow \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{mod } 2).$$
The theta group

We work with a slightly different group $\Gamma(2)$. The theta group, $\Gamma_\theta$, is the group generated by the Möbius transformations

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$\Gamma(2)$ is an index two subgroup of $\Gamma_\theta$, which is an index three subgroup of the modular group and hence a Fuchsian group.
The theta-surface

The region $D$ is a fundamental domain for the action of $\Gamma_\theta$ on $\mathbb{H}$.

The quotient $\mathbb{H}/\Gamma_\theta$ is a sphere with two cusps, at $\infty$ and 1, and a cone point at $i$. 
Continued fractions and the theta group

Elements of the theta group relate naturally to ECFs. Since $T$ is an involution, any element can be written

$$S^{b_1} TS^{b_2} T \cdots S^{b_k} TS^{b_{k+1}},$$

with $b_i \in \mathbb{Z}$ and $b_i \neq 0$ for $i \neq 1, k + 1$. Since

$$S^n(z) = 2n + z,$$

we have

$$S^{b_1} TS^{b_2} T \cdots S^{b_k} TS^{b_{k+1}}(\infty) = 2b_1 - \frac{1}{2b_2 - \frac{1}{2b_3 - \cdots - \frac{1}{2b_k}}}.$$
Forming the theta graph

Let $L$ denote the hyperbolic geodesic joining 0 and $\infty$ in the upper half-plane $\mathbb{H}$. 

\[ \infty \]

\[ \downarrow \]

\[ 0 \]
Forming the theta graph

**Definition**

The *theta graph* $G_\theta$ is the graph obtained as the orbit of $L$ under elements of the theta group.
• Vertices of $G_\theta$ are the set $P$ of reduced rationals $\frac{p}{q}$ with either $p$ odd and $q$ even, or $p$ even and $q$ odd, and $\infty$. 
The theta graph

- Vertices of $G_\theta$ are the set $P$ of reduced rationals $\frac{p}{q}$ with either $p$ odd and $q$ even, or $p$ even and $q$ odd, and $\infty$.
- $G_\theta$ is an infinite tree.
The theta graph

- Vertices of $G_\theta$ are the set $P$ of reduced rationals $\frac{p}{q}$ with either $p$ odd and $q$ even, or $p$ even and $q$ odd, and $\infty$.
- $G_\theta$ is an infinite tree.

Vertices surrounding a ‘face’ accumulate at a face-centre.

- Face-centres are the reduced rationals $\frac{p}{q}$ with both $p$ and $q$ odd.
The Farey graph

The theta graph can be viewed as a subgraph of the Farey graph, the orbit of $L$ under $\Gamma$.

The vertices that are ‘missing’ from $G_\theta$ are the face-centre points.

The Farey graph can be used to study integer continued fractions (i.e. Series [4], and Beardon, Hockman, Short [1]).
Recall that

\[ [2b_1, 2b_2, \ldots, 2b_n] = S^{b_1} T S^{b_2} T \cdots T S^{b_n} T(\infty). \]
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so the vertices of \( G_\theta \) correspond to finite ECFs.
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so the vertices of $G_\theta$ correspond to finite ECFs.

The convergents of an ECF $[2b_1, 2b_2, \ldots]$ are the numbers $C_k = [2b_1, 2b_2, \ldots, 2b_k]$. Each convergent is a vertex of $G_\theta$. 
Recall that

\[ [2b_1, 2b_2, \ldots, 2b_n] = S^{b_1} T S^{b_2} T \ldots T S^{b_n} T(\infty). \]

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The convergents of an ECF \([2b_1, 2b_2, \ldots]\) are the numbers \( C_k = [2b_1, 2b_2, \ldots, 2b_k] \). Each convergent is a vertex of \( G_\theta \).

**Theorem**

*The vertices \( \infty = v_1, v_2, v_3, \ldots \) form a simple path in \( G_\theta \) if and only if they are the consecutive convergents of an ECF expansion.*
Paths in the theta graph

For example, take \([0, -2, -2, 2, \ldots]\).

\[
C_1 = 0, \quad C_2 = \frac{1}{2}, \quad C_3 = \frac{2}{3}, \quad C_4 = \frac{5}{8}, \ldots
\]
Basic properties of even-integer continued fractions

Properties of the graph can be used to show that:

- Every infinite ECF converges.
- Every real number $x$ has an ECF expansion, which is finite if and only if $x = \frac{p}{q}$ is a reduced rational with either $p$ odd and $q$ even, or $p$ even and $q$ odd.
- The ECF expansion of a real number $x$ is unique unless $x = \frac{p}{q}$ is a reduced rational with both $p$ and $q$ odd.
- The ECF expansion of a real number is given by the nearest even-integer algorithm.

What about Diophantine approximation?
Strong approximation

Recall that a rational number $x = \frac{a}{b} \in P$ is a \textit{theta-strong approximant} for an irrational number $\alpha$ if for any other rational number $\frac{c}{d} \in P$ with $d \leq b$,

$$|b\alpha - a| \leq |d\alpha - c|,$$

with equality if and only if $\frac{a}{b} = \frac{c}{d}$.

We want to know how the theta-strong approximants relate to the ECF convergents.

The structure of the graph shows that the convergents $C_k$ of the ECF expansion of $\alpha$ become arbitrarily close to $\alpha$, but it doesn’t say anything about exactly how close each convergent is.

We need to introduce another geometric tool: Ford circles.
Ford circles

Definition

The *Ford circle* $C_x$ of a rational number $x = \frac{a}{b}$ is the horocycle with radius $\frac{1}{2b^2}$ and base point $x$.

Ford circles were introduced by Ford nearly 100 years ago (see [2]), and have since been used by many to study continued fractions.
Ford circles

We place a Ford circle at each rational number.
Ford circles and the Farey graph

Ford circles are closely related to the Farey graph: Two Ford circles are tangent if and only if their basepoints are joined by an edge in the Farey graph.
Ford circles and the theta graph

We can restrict the collection of Ford circles to those based at points in $P$. This gives a collection corresponding to the theta graph.

Paths in the theta graph correspond to chains of Ford circles. Knowing the radii of the circles allows us to study approximation properties.
Given an irrational number $\alpha$, and a point $x = \frac{a}{b} \in P$, define

$$R_x(\alpha) = \frac{1}{2} |b\alpha - a|^2$$

$R_x(\alpha)$ is the radius of the horocycle based at $\alpha$, tangent to $C_x$. 
Reformulating the problem

Given \( x = \frac{a}{b} \) and \( y = \frac{c}{d} \), notice that \( d \leq b \) if and only if the radius of \( C_y \) is greater than or equal to the radius of \( C_x \).

Clearly,

\[
|b\alpha - a| \leq |d\alpha - c| \iff \frac{1}{2}|b\alpha - a|^2 \leq \frac{1}{2}|d\alpha - c|^2
\]

\[
\iff R_x(\alpha) \leq R_y(\alpha)
\]

\( x \) is a theta-strong approximant for \( \alpha \) if and only if for any \( y \) with \( \text{Rad}(C_x) \leq \text{Rad}(C_y) \), \( R_x(\alpha) \leq R_y(\alpha) \)
Reformulating the problem

This says that $x$ is a theta-strong approximant if and only if for any $y$ with $C_y$ larger than $C_x$, a horocycle based at $\alpha$ that is expanding in size will hit $C_x$ before it hits $C_y$. 

\[\text{Diagram showing Ford circles and horocycles.}\]
Convergents are theta-strong approximants

**Theorem**

* A rational number is a theta-strong approximant for an irrational number \( \alpha \) if and only if it is a convergent of the ECF expansion of \( \alpha \).

1. If \( \text{Rad}(C_y) \geq \text{Rad}(C_x) \) then \( y \) lies outside \([x, c']\).
2. Then \( R_x(\alpha) \leq R_y(\alpha) \) and \( x \) is a theta-strong approximant.
**Theta-strong approximants are convergents**

Given a non-convergent $x = \frac{a}{b}$, there are convergents $v_i = \frac{p_i}{q_i}$ and $v_{i+1} = \frac{p_{i+1}}{q_{i+1}}$ with $q_i \leq b < q_{i+1}$. Then

$$\text{Rad}(C_{v_{i+1}}) < \text{Rad}(C_x) \leq \text{Rad}(C_{v_i}).$$

1. Since $\text{Rad}(C_x) > \text{Rad}(C_{v_{i+1}})$, $x$ lies outside $[v_i, c']$.
2. Then $R_{v_i}(\alpha) \leq R_x(\alpha)$, so $x$ is not a theta-strong approximant.
The general case

Both strong approximation and theta-strong approximation are examples of a more general problem.

Given a Fuchsian group $G$, with an equivalence class of parabolic fixed points $P$ (that contains $\infty$), how well can a point in $\hat{\mathbb{R}} \setminus P$ be approximated by points in $P$?
The general case

Both strong approximation and theta-strong approximation are examples of a more general problem.

Given a Fuchsian group $G$, with an equivalence class of parabolic fixed points $P$ (that contains $\infty$), how well can a point in $\hat{\mathbb{R}} \setminus P$ be approximated by points in $P$?

**Definition**

A number $\frac{a}{b} \in P$ is a $G$-strong approximant for a number $\alpha \in \hat{\mathbb{R}} \setminus P$ if for any other $\frac{c}{d} \in P$ with $d \leq b$,

$$|b\alpha - a| \leq |d\alpha - c|,$$

with equality if and only if $\frac{a}{b} = \frac{c}{d}$.
The geometric reformulation

To get the Ford circle definition of $G$-strong approximant, we place horocycles of radius $\frac{k}{b^2}$ at each parabolic fixed point. Increasing $k$ increases the radius of the horocycles.

The idea is to find some $k$ for which the horocycles form appropriate chains, so that the $G$-strong approximants can be determined.
Back to $\Gamma(2)$

In the case of $\Gamma(2)$, we have three equivalence classes of parabolic fixed points. Consider the orbit of $\infty$ - the rational numbers $\frac{p}{q}$ with $p$ odd and $q$ even. Setting $k = 1$ gives the following picture.

The Ford circles can be mapped to those of $\Gamma_\theta$ by the map $z \rightarrow 2z + 1$, so approximation by rational numbers $\frac{p}{q}$ with $p$ odd and $q$ even can be determined from the $\Gamma_\theta$ case.
Thanks for listening

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